7 LIMITS

Let's Study

- Meaning – Definition of Limit
- Calculation of various limits
- Limits of Trigonometric Functions
- Limits of Exponential and Logarithmic Functions
- Limit at Infinity and Infinite limit

Introduction:

Calculus is an important branch of mathematics. The concept of limit of a function is a fundamental concept in calculus.

7.1.1 Definition of Limit:

Let's :Learn

7.1.1 LIMIT OF A FUNCTION :

Suppose x is a variable and a is a constant. If x takes values closer and closer to 'a' but not equal to 'a', then we say that x tends to a. Symbolically it is denoted as $x \to a$.

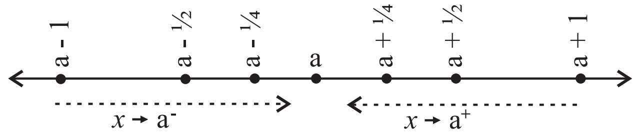

Fig. 7.1

You can observe that the values of x are very near to 'a' but not equal to 'a'.

When $x > a$ and x takes values near a, for example, $x = a + \frac{1}{8}, x = a + \frac{1}{4}, x = a + \frac{1}{2}$ ... etc; we say that $x \to a+$ (x tends to a from larger values). When $x < a$ and x takes values near a, for example, $x = a - \frac{1}{2}, x = a - \frac{1}{4}, x = a - \frac{1}{8}$ ... etc. then we say that $x \to a^-$ (x tends to a from smaller values).

We will study functions of x, a real variable, and a, b, c etc will denote constants. $x \to a$ implies that x takes values as near a as possible. So in this case we have to consider x going nearer a from either side. So, $x = a - \frac{1}{4}, a + \frac{1}{4}, a - \frac{1}{2}, a + \frac{1}{2}$ ......

We will illustrate with an example.

Consider the function $f(x) = x + 3$

Take the value of x very close to 3 but not equal to 3.

The following table shows that as x gets nearer to 3, the corresponding values of $f(x)$ also get nearer to 6.

(I)

	x approaches to 3 from left				
x	2.5	2.6	…	2.9	2.99
$f(x)$	5.5	5.6	…	5.9	5.99

(II)

	x approaches to 3 from right				
x	3.6	3.5	…	3.1	3.01
$f(x)$	6.6	6.5	…	6.1	6.01

From the table we see that as $x \to 3$ from either side, $f(x) \to 6$.

This idea can be expressed by saying that the limiting value of $f(x)$ is 6 when x approaches 3, at $x = 3$, which is the limiting value of $f(x)$ as $x \to 3$

Observe that if $P(x)$ is a polynomical in x, then $\lim_{x \to a} P(x) = P(a)$, for any constant '$a$'.

We are going to study the limit of a rational function $f(x) = \frac{P(x)}{Q(x)}$ as $x \to$ a.

Here $P(x)$ and $Q(x)$ are polynomials in x.

We get three different cases.

(1) $Q(a) \neq 0$,

(2) $Q(a) = 0$ and $P(a) = 0$

(3) $Q(a) = 0$ and $P(a) \neq 0$

In **case (1)** the limit of $f(x)$ as $x \to a$ is

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

In **Case (2)** $(x - a)$ is a factor of $P(x)$ as well as $Q(x)$. So we expresse $P(x)$ and $Q(x)$ as $P(x) = (x - a)P_1(x)$ and $Q(x) = (x - a)Q_1(x)$

$$\text{Now} = \frac{P(x)}{Q(x)} = \frac{(x-a)^r P_1(x)}{(x-a)^s Q_1(x)} \quad (x - a \neq 0)$$

If r = s then $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P_1(a)}{Q_1(a)}$

If r > s then $\lim_{x \to a} \frac{P(x)}{Q(x)} = 0$

If r < s then we proceed to case (3).

In **case (3),** if $Q(a) = 0$ and $P(a) \neq 0$

$\lim_{x \to a} \frac{P(x)}{Q(x)}$ does not exist.

7.1.2 Definition of Limit :

We need to confirm that $f(x)$ is very near to l (or as near as expected). This is expressed by $|f(x) - l| < \in$ for any $\in > 0$. Here $\in$ can be arbitrarily small to ensure that f(x) is very near l. If this condition is satisfied for all x near enough, then we can say that $f(x) \to l$ as $x \to a$, the fact that x is near enough a is expressed by $0 < |x - a| < \delta$ where $\delta > 0$. This δ can be very small and depends upon f(x) and $\in$.

Hence the Definition

> *If given $\in > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \in$ for all $|x - a| < \delta$,* then we say that *$f(x) \to l$ as $x \to a$.*

SOLVED EXAMPLES

Strategy : Steps for verifying the $\in$ - δ definition.

Consider $\in > 0$ given, substitute the values of $f(x)$ and l in $|f(x) - l| < \in$ and proceed to find the value of δ. We may have to manipulate the inequalities.

Ex. 1. Consider the example $f(x) = 3x + 1$, take $a = 0$ and $l = 1$

We want to find some $\delta > 0$ such that,

$0 < |x - 0| < \delta$ impies that, $|(3x + 1) - 1| < \in$

if $|3x| < \in$ i.e. if $3|x| < \in$ i.e. if $|x| < \frac{\in}{3}$

So, we can choose $\delta = \frac{\in}{3}$

(If fact any $\delta \leq \frac{\in}{3}$ will do.)

$\therefore\ 0 < |x - 0| < \delta \Rightarrow |f(x) - l| < \in$

$\therefore\ \lim_{x \to 0} (3x + 1) = 1$

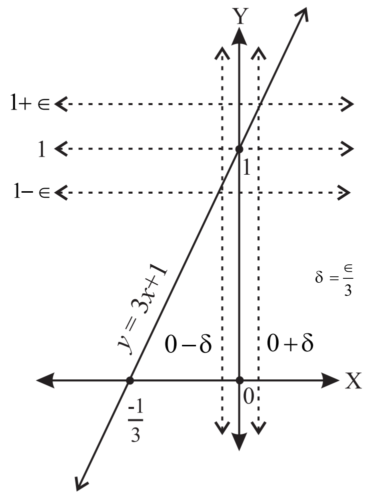

Fig. 7.2

Ex. 2. $f(x) = x^2$

Here take $a = 3$ and $l = 9$

We want to find some $\delta > 0$ such that

$0 < |x - 3| < \delta$ implies $|x^2 - 9| < \in$

i.e. $3 - \delta < x < 3 + \delta \Rightarrow |x^2 - 9| < \in$

δ can be chosen as small as we like and take $\delta < 1$

Then, $3 - \delta < x < 3 + \delta \Rightarrow 2 < x < 4$ or $5 < x + 3 < 7$

We want $|(x + 3)(x - 3)| < \in$

But $|(x + 3)(x - 3)| < 7|x - 3|$

So $7|(x - 3)| < \in \Rightarrow |x^2 - 9| < \in$

If $\delta = \frac{\in}{7}$, $|(x - 3)| < \delta \Rightarrow |x^2 - 9| < \in$

So we choose $\delta = \min\{\frac{\in}{7}, 1\}$

then $|(x - 3)| < \delta \Rightarrow |f(x) - 9| < \in$

* Note that, we want to get rid of factor $|x + 3|$ Hence we have to get its lower bound.

Ex. 3. $f(x) = [x]$, $2 < x < 4$ where $[x]$ is a greatest integer function.

We have seen the $f(x) = [x]$, $2 \le x \le 4$

Note that

$[x] = 2$ for $2 \le x < 3$

$= 3$ for $3 \le x < 4$

Let us study the limits of $f(x)$ as $x \to 3$ and $x \to 2.7$

$\lim_{x \to 3^+} f(x) = 3$,

But for $x < 3$, $f(x) = 2$. So, $\lim_{x \to 3^-} f(x) = 2$.

If we take $l = 3$, then for $\in = \frac{1}{2}$ and any $\delta > 0$

$3 - \delta < x < 3 \Rightarrow f(x) = 2$ and $|f(x) - l| = 1 \nless \in$

If we take $l = 2$, then $3 < x < 3 + \delta \Rightarrow f(x) = 3$,

$|f(x) - 2| = 1 \nless \in$

$\therefore \lim_{x \to 3} f(x)$ does not exist.

Consider $\lim_{x \to 2.7} f(x)$

Consider $a = 2.7$ We see that for $2 < x < 3$, $f(x) = 2$.

If we choose $\delta = 0.3$,

then $2 < 2.7 - \delta < 2.7 < 2.7 + \delta < 3$

and $f(x) = 2$ is a constant.

$\therefore \lim_{x \to 2.7} f(x) = 2$

From the above example we notice that the limits of $f(x)$ as $x \to a^+$ or a^- can be different. This induces us to define the following.

7.1.3 One Sided Limit: $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$; if they exist are called one sided limits.

7.1.4 Left hand Limit: *If given $\in > 0$, there exists $\delta > 0$ such that for $|f(x) - l| < \in$ for all x with $a - \delta < x < a$ then*

$\lim_{x \to a^-} f(x) = l$

7.1.5 Right hand Limit : *If given $\in > 0$ there exists $\delta > 0$ such that for $|f(x) - l| < \in$ for all x with $a < x < a + \delta$ then*

$\lim_{x \to a^+} f(x) = l$

7.1.6 Existence of a limit of a function at a point $x = a$

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$, then limit of the function $f(x)$ as $x \to a$ exists and its value is l. And if $\lim_{x \to a^+} f(x) \ne \lim_{x \to a^-} f(x)$ then $\lim_{x \to a} f(x)$ does not exist.

Example:

Find left hand limit and right hand limit for the following function.

$$f(x) = \begin{cases} 3x + 1 & \text{if } x < 1 \\ 7x^2 - 3 & \text{if } x \ge 1 \end{cases}$$

Solution : Right hand limit, $\lim_{x \to 1} f(x)$, for $x > 1$

i.e. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (7x^2 - 3) = 4$

Left hand limit, $\lim_{x \to 1} f(x)$, for $x < 1$:

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x + 1) = 4$

Since left and right-hand limits are equal, the two-sided limit is defined, and $\lim_{x \to 1} f(x) = 4$.

Note : $\lim_{x \to a} f(x)$ means $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

7.1.7 ALGEBRA OF LIMITS:

It is easy to verify the following.

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$= l \pm m$

2. $\lim_{x\to a} [f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
$= l \times m$

3. $\lim_{x\to a} [k\, f(x)] = k \times \lim_{x\to a} f(x) = kl$, where 'k' is a constant

4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l}{m}$ where $m \neq 0$

Note:

1) $\lim_{x\to a} k = k$, where k is a constant

2) $\lim_{x\to a} x = a$

3) $\lim_{x\to a} x^n = a^n$

4) $\lim_{x\to a} \sqrt[r]{x} = \sqrt[r]{a}$

5) If p(x) is a polynomial,

then $\lim_{x\to a} p(x) = p(a)$

While evaluating limits, we must always check whether the denominator tends to zero, and if it does, then whether the numerator also tends to zero. In case both tend to zero we have to study the function in detail.

SOLVED EXAMPLES

Ex. 1 : $\lim_{n\to 3}\left(\sum_{r=1}^{n} r^2\right) = \lim_{n\to 3}\left(\dfrac{n(n+1)(2n+1)}{6}\right)$

$= \dfrac{(3)\times(3+1)\times(6+1)}{6}$

$= \dfrac{(3)\times(3+1)\times(6+1)}{6}$

$= \dfrac{3\times 4\times 7}{6}$

$= 14$

Ex. 2 : $\lim_{y\to 2} [(y^2-3)(y+2)]$

$= \lim_{y\to 2} [(y^2-3)\,[y+2]$

$= (2^2-3)(2+2) = (8-3)(4) = 5 \times 4 = 20$

Ex. 3 : $\lim_{x\to 3}\left(\dfrac{\sqrt{6+x}-\sqrt{7-x}}{x}\right)$

$= \dfrac{\lim_{x\to 3}\left(\sqrt{6+x}\right) - \lim_{x\to 3}\left(\sqrt{7-x}\right)}{\lim_{x\to 3}(x)}$

$= \dfrac{\sqrt{6+3}-\sqrt{7-3}}{3}$

$= \dfrac{\sqrt{9}-\sqrt{4}}{3} = \dfrac{3-2}{3} = \dfrac{1}{2}$

Ex. 4 : $\lim_{x\to 1}\left(\dfrac{\frac{1}{x}-1}{x-1}\right)$

$= \lim_{x\to 1}\left(\dfrac{\frac{1-x}{x}}{x-1}\right)$

$= \lim_{x\to 1}\left[\dfrac{(1-x)}{(x-1)\times x}\right]$

$= \lim_{x\to 1}\left[\dfrac{-(x-1)}{(x-1)\times x}\right]$

$= \lim_{x\to 1}\left[\dfrac{-1}{x}\right]$ [As $x \to 1$, $x-1 \neq 0$]

$= -\lim_{x\to 1}\left[\dfrac{1}{x}\right] = -\dfrac{1}{1}$

$= -1$

Ex. 5 : Discuss the limit of the following function as x tends to 3 if

$$f(x) = \begin{cases} x^2+x+1, & 2 \le x \le 3 \\ 2x+1, & 3 < x \le 4 \end{cases}$$

Solution: As $f(x)$ is defined seperately for $x \le 3$ and $x > 3$, we have to find left hand limit (when $x \le 3$) and right hand limit (when $x > 3$) to discuss the existance of limit of $f(x)$ as $x \to 3$.

For the interval $2 \le x \le 3$; $f(x) = x^2 + x + 1$

$$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 + x + 1) = (3)^2 + 3 + 1$$

$$= 9 + 3 + 1 = 13 \qquad \text{-----(I)}$$

Similarly for the interval $3 < x \le 4$; $f(x) = 2x + 1$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x + 1) = (2 \times 3) + 1$$

$$= 6 + 1 = 7 \qquad \text{-----(II)}$$

From (I) and (II), $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Ex. 6 : For a given $\in > 0$, find $\delta > 0$ such that whenever $|x - a| < \delta$, we have $|f(x) - l| < \in$ so that $\lim_{x\to 1} (4x + 3) = 7$

Solution : We want to find δ so that $\lim_{x\to 1} (4x + 3) = 7$

Here $a = 1$, $l = 7$ and $f(x) = 4x + 3$

Consider $\in > 0$ and $|f(x) - l| < \in$

$$|(4x + 3) - 7| < \in$$

$$\text{if } |(4x - 4)| < \in$$

$$\text{i.e. if } |4(x - 1)| < \in$$

$$\text{i.e. if } |x - 1| < \frac{\varepsilon}{4}$$

We can have $\delta \le \frac{\varepsilon}{4}$ so that $|(x - 1)| < \delta \Rightarrow |f(x) - 7| < \in$

77.1.8 Theorem: Prove that

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1} \text{ where } n \in \mathbb{N},\ a > 0$$

Proof : We know, $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$ for $n \in \mathbb{N}$.

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) =$$

$$\lim_{x\to a}\left(\frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots + a^{n-1})}{x-a}\right)$$

$$= \lim_{x\to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$$

for $[x - a \neq 0]$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

Note : The above limit can also be found by using the substitution $x - a = h$.

$x - a = h \therefore x = a + h$ and $x \to a \Rightarrow h \to 0$. Use binomial theorem to expand $(a + h)^n$, simplify and apply the limit to get the result

$$\lim_{x\to a}\left[\frac{x^n - a^n}{x - a}\right] = n\, a^{n-1}$$

Verify : If $n < 0$ say $n = -m$ then

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) = \lim_{x\to a}\left(\frac{x^{-m} - a^{-m}}{x - a}\right)$$

$$= -ma^{-m-1}$$

Note : The above theorem can also be verified if n is a fraction say $n = \frac{p}{q}$ where $q \neq 0$. Then

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x - a}\right) = \lim_{x\to a}\left(\frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a}\right) = \frac{p}{q}\, a^{\frac{p}{q}-1}$$

SOLVED EXAMPLES

Ex. 1 : Evaluate $\lim_{x\to 5}\left(\frac{x^4-625}{x-5}\right)$

Solution : $\lim_{x\to 5}\left(\frac{x^4-625}{x-5}\right) = \lim_{x\to 5}\left(\frac{x^4-5^4}{x-5}\right)$

$= 4(5)^{4-1}$..... $\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$= 4(125) = 500$

Ex. 2 : Evaluate $\lim_{x\to 2}\left(\frac{x^7-128}{x^5-32}\right)$

Solution : $\lim_{x\to 2}\left(\frac{x^7-128}{x^5-32}\right) = \lim_{x\to 2}\left(\frac{x^7-2^7}{x^5-2^5}\right)$

$= \lim_{x\to 5}\left(\frac{\frac{x^7-2^7}{x-2}}{\frac{x^5-2^5}{x-2}}\right)$

... [As $x\to 2$, $x-2\neq 0$]

$= \frac{7(2)^6}{5(2)^4}$

...$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$= \frac{7(2)^2}{5}$

$= \frac{28}{5}$

Ex. 3 : If $\lim_{x\to 4}\left[\frac{x^n-4^n}{x-4}\right] = 48$ and $n\in N$, find n.

Solution : Given $\lim_{x\to 4}\left[\frac{x^n-4^n}{x-4}\right] = 48$

$\therefore n(4)^{n-1} = 48$

$\therefore n(4)^{n-1} = 3(4)^{3-1}$

$\therefore n = 3$ .... by comparing

Ex. 4 : Evaluate $\lim_{x\to 1}\left[\frac{2x-2}{\sqrt[3]{26+x}-3}\right]$

Solution: Put $26 + x = t^3$, $\therefore x = t^3 - 26$

As $x\to 1$, $t\to 3$

$\therefore \lim_{x\to 1}\left[\frac{2x-2}{\sqrt[3]{26+x}-3}\right]$

$= \lim_{t\to 3}\left[\frac{2\left(t^3-26\right)-2}{\sqrt[3]{t^3}-3}\right]$

$= \lim_{x\to 3}\left[\frac{2\left(t^3-3^3\right)}{t-3}\right]$

$= 2\lim_{t\to 3}\left[\frac{t^3-3^3}{t-3}\right]$

$= 2\times 3(3)^{3-1}$

$= 54$

EXERCISE 7.1

Q.I Evaluate the following limits :

1. $\lim_{z\to -3}\left[\frac{\sqrt{z+6}}{z}\right]$

2. $\lim_{y\to -3}\left[\frac{y^5+243}{y^3+27}\right]$

3. $\lim_{z\to -5}\left[\frac{\left(\frac{1}{z}+\frac{1}{5}\right)}{z+5}\right]$

Q.II Evaluate the following limits :

1. $= \lim_{x\to 3}\left[\frac{\sqrt{2x+6}}{x}\right]$

2. $\lim_{x\to 2}\left[\frac{x^{-3}-2^{-3}}{x-2}\right]$

3. $\lim_{x\to 5}\left[\frac{x^3-125}{x^5-3125}\right]$

4. If $\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right]=\lim_{x\to a}\left[\frac{x^3-a^3}{x-a}\right]$,

find all possible values of a.

Q.III Evaluate the following limits :

1. $\lim_{x\to 1}\left[\frac{x+x^2+x^3+........+x^n-n}{x-1}\right]$

2. $\lim_{x\to 7}\left[\frac{\left(\sqrt[3]{x}-\sqrt[3]{7}\right)\left(\sqrt[3]{x}+\sqrt[3]{7}\right)}{x-7}\right]$

3. If $\lim_{x\to 5}\left[\frac{x^k-5^k}{x-5}\right]=500$, find all possible values of k.

4. $\lim_{x\to 0}\left[\frac{(1-x)^8-1}{(1-x)^2-1}\right]$

5. $\lim_{x\to 0}\left[\frac{\sqrt[3]{1+x}-\sqrt{1+x}}{x}\right]$

6. $\lim_{y\to 1}\left[\frac{2y-2}{\sqrt[3]{7+y}-2}\right]$

7. $\lim_{z\to a}\left[\frac{(z+2)^{\frac{3}{2}}-(a+2)^{\frac{3}{2}}}{z-a}\right]$

8. $\lim_{x\to 7}\left[\frac{x^3-343}{\sqrt{x}-\sqrt{7}}\right]$

9. $\lim_{x\to 1}\left(\frac{x+x^3+x^5+...+x^{2n-1}-n}{x-1}\right)$

Q.IV In the following examples, given $\in\ > 0$, find a $\delta > 0$ such that whenever, $|x-a| < \delta$, we must have $|f(x)-l| < \in$

1. $\lim_{x\to 2}(2x+3)=7$

2. $\lim_{x\to -3}(3x+2)=-7$

3. $\lim_{x\to 2}(x^2-1)=3$

4. $\lim_{x\to 1}(x^2+x+1)=3$

7.2 METHOD OF FACTORIZATION:

$P(x)$ and $Q(x)$ are polynomials in x such that

$f(x) = \frac{P(x)}{Q(x)}$. We consider $\lim_{x\to a} f(x)$.

Let's check $\lim_{x\to a} Q(x)$ and $\lim_{x\to a} P(x)$.

1) If $\lim_{x\to a} Q(x) = m \neq 0$

then $\lim_{x\to a} f(x) = \lim_{x\to a}\left[\frac{P(x)}{m}\right]$

2) If $\lim_{x\to a} Q(x) = 0$, then $(x-a)$ divides $Q(x)$. In such a case if $(x-a)$ does not divide $P(x)$ then $\lim_{x\to a} f(x)$ does not exist.

(3) Further if $\lim_{x\to a} P(x)$ is also 0, then $(x-a)$ is a factor of both $P(x)$ and $Q(x)$.

So $\lim_{x\to a} f(x) = \lim_{x\to a}\left[\frac{P(x)/(x-a)}{Q(x)/(x-a)}\right]$.

Factorization of polynomials is a useful tool to determine the limits of rational algebraic expressions.

SOLVED EXAMPLES

Ex.1 : Evaluate $\lim_{z\to 3}\left[\frac{z(2z-3)-9}{z^2-4z+3}\right]$

Solution: If we substitute $z = 3$ in numerator and denominator, we get z(2z – 3) – 9 = 0 and $z^2 - 4z + 3 = 0$

So (z – 3) is a factor in the numerator and denominator.

$$\lim_{z\to3}\left[\frac{z(2z-3)-9}{z^2-4z+3}\right] = \lim_{z\to3}\left[\frac{2z^2-3z-9}{z^2-4z+3}\right]$$

$$= \lim_{z\to3}\left[\frac{(z-3)(2z+3)}{(z-3)(z-1)}\right]$$

$$= \lim_{z\to3}\left[\frac{(2z+3)}{(z-1)}\right]$$

...[As $z\to3$, $z - 3 \neq 0$]

$$= \frac{2(3)+3}{3-1}$$

$$= \frac{9}{2}$$

Ex.2: Evaluate $\lim_{x\to4}\left[\frac{(x^3-8x^2+16x)^9}{(x^2-x-12)^{18}}\right]$

Solution : $\lim_{x\to4}\left[\frac{\left[x(x-4)^2\right]^9}{(x-4)^{18}(x+3)^{18}}\right]$

$$= \lim_{x\to4}\left[\frac{(x-4)^{18}x^9}{(x-4)^{18}(x+3)^{18}}\right]$$

$$= \lim_{x\to4}\left[\frac{x^9}{(x+3)^{18}}\right]$$

...[As $x\to4$, $x - 4 \neq 0$]

$$= \frac{4^9}{7^{18}}$$

Ex.3: Evaluate $\lim_{x\to1}\left[\frac{1}{x-1}+\frac{2}{1-x^2}\right]$

Solution : $\lim_{x\to1}\left[\frac{1}{x-1}+\frac{2}{(1-x)(x+1)}\right]$

$$= \lim_{x\to1}\left[\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to1}\left[\frac{1+x-2}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to1}\left[\frac{x-1}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to1}\left[\frac{1}{(x+1)}\right]$$

...[As $x\to1$, $x - 1 \neq 0$]

$$= \frac{1}{2}$$

Ex.4: Evaluate $\lim_{x\to1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$

Solution : In this case (x – 1) is a factor of the numerator and denominator.

To find another factor we use synthetic division.

Numerator: $x^3 + x^2 - 5x + 3$

1	1	1	–5	3
		1	2	–3
	1	2	–3	0

$\therefore x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3)$

Denominator: $x^2 - 1 = (x + 1)(x - 1)$

$$\lim_{x\to1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$$

$$= \lim_{x\to1}\left[\frac{(x-1)(x^2+2x-3)}{(x+1)(x-1)}\right]$$

$$= \lim_{x\to1}\left[\frac{x^2+2x-3}{x+1}\right]$$ ...[As $x\to1$, $x - 1 \neq 0$]

$$= \frac{1+2-3}{1+1} = 0$$

Ex.5: Evaluate $\lim_{x\to 0}\left(\frac{1-\sqrt[3]{x^2+1}}{x^2}\right)$

Solution : Put $\sqrt[3]{x^2+1}$ = t, $x^2 + 1 = t^3$

$\therefore$ $x^2 = t^3 - 1$, as $x\to 0$, $t\to 1$

$$\lim_{x\to 0}\left(\frac{1-\sqrt[3]{x^2+1}}{x^2}\right) = \lim_{t\to 1}\left(\frac{(1-t)}{(t^3-1)}\right)$$

$$= \lim_{t\to 1}\left(\frac{-(t-1)}{(t-1)(t^2+t+1)}\right)$$

$$= \lim_{t\to 1}\left(\frac{-1}{t^2+t+1}\right)$$

...[As $t\to 1$, $t-1 \neq 0$]

$$= \frac{-1}{1+1+1} = -\frac{1}{3}$$

EXERCISE 7.2

Q.I Evaluate the following limits :

1. $\lim_{z\to 2}\left[\frac{z^2-5z+6}{z^2-4}\right]$

2. $\lim_{x\to -3}\left[\frac{x+3}{x^2+4x+3}\right]$

3. $\lim_{y\to 0}\left[\frac{5y^3+8y^2}{3y^4-16y^2}\right]$

4. $\lim_{x\to -2}\left[\frac{-2x-4}{x^3+2x^2}\right]$

5. $\lim_{x\to 3}\left[\frac{x^2+2x-15}{x^2-5x+6}\right]$

Q.II Evaluate the following limits :

1. $\lim_{u\to 1}\left[\frac{u^4-1}{u^3-1}\right]$

2. $\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{9x}{x^3-27}\right]$

3. $\lim_{x\to 2}\left[\frac{x^3-4x^2+4x}{x^2-1}\right]$

4. $\lim_{\Delta x\to 0}\left[\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}\right]$

5. $\lim_{x\to\sqrt{2}}\left[\frac{x^2+x\sqrt{2}-4}{x^2-3x\sqrt{2}+4}\right]$

6. $\lim_{x\to 2}\left[\frac{x^3-7x+6}{x^3-7x^2+16x-12}\right]$

Q.III Evaluate the Following limits :

1. $\lim_{y\to \frac{1}{2}}\left[\frac{1-8y^3}{y-4y^3}\right]$

2. $\lim_{x\to 1}\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

3. $\lim_{x\to 1}\left[\frac{x^4-3x^2+2}{x^3-5x^2+3x+1}\right]$

4. $\lim_{x\to 1}\left[\frac{x+2}{x^2-5x+4}+\frac{x-4}{3(x^2-3x+2)}\right]$

5. $\lim_{x\to a}\left[\frac{1}{x^2-3ax+2a^2}+\frac{1}{2x^2-3ax+a^2}\right]$

7.3 METHOD OF RATIONALIZATION:

If the function in the limit involves a square root or a trigonometric function, it may be possible to simplify the expression by multiplying and dividing by its rationalizing factor.

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

Soluton : $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

$= \lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)$

$= \lim_{x\to 0}\left(\frac{1+x-1}{x\left(\sqrt{1+x}+1\right)}\right)$

$= \lim_{x\to 0}\left(\frac{x}{x\left(\sqrt{1+x}+1\right)}\right)$

$= \lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}+1}\right)$ ...[As $x\to 0$, $x\neq 0$]

$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$

Ex. 2. Evaluate $\lim_{z\to 0}\left[\frac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

Solution : $\lim_{z\to 0}\left[\frac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

$= \lim_{z\to 0}\left[\frac{\sqrt{b+z}-\sqrt{b-z}}{z}\right]$

$= \lim_{z\to 0}\left[\frac{\sqrt{b+z}-\sqrt{b-z}}{z}\times\frac{\sqrt{b+z}+\sqrt{b-z}}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\frac{(b+z)-(b-z)}{z}\times\frac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\frac{2z}{z}\times\frac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$

$= \lim_{z\to 0}\left[\frac{2}{\sqrt{b+z}+\sqrt{b-z}}\right]$ ...[As $z\to 0$, $z\neq 0$]

$= \frac{2}{\sqrt{b+0}+\sqrt{b-0}}$

$= \frac{2}{2\sqrt{b}}$

$= \frac{1}{\sqrt{b}}$

Ex. 3. Evaluate $\lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\right)$

Solution : $\lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\right)$

$= \lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\times\frac{\sqrt{x^2-7}+\sqrt{25-x^2}}{\sqrt{x^2-7}+\sqrt{25-x^2}}\right)$

$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{x^2-7-25+x^2}\right)$

$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x^2-16)}\right)$

$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x-4)(x+4)}\right)$

$$= \lim_{x\to 4}\left(\frac{(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x+4)}\right)$$

...[As $x\to 4$, $x - 4 \neq 0$]

$$= \frac{(4+5)\left(\sqrt{4^2-7}+\sqrt{25-4^2}\right)}{2(4+4)} = \frac{(9)(3+3)}{2(8)} = \frac{27}{8}$$

EXERCISE 7.3

Q.I Evaluate the following limits :

1. $\lim_{x\to 0}\left[\frac{\sqrt{6+x+x^2}-\sqrt{6}}{x}\right]$

2. $\lim_{x\to 3}\left[\frac{\sqrt{2x+3}-\sqrt{4x-3}}{x^2-9}\right]$

3. $\lim_{y\to 0}\left[\frac{\sqrt{1-y^2}-\sqrt{1+y^2}}{y^2}\right]$

4. $\lim_{x\to 2}\left[\frac{\sqrt{2+x}-\sqrt{6-x}}{\sqrt{x}-\sqrt{2}}\right]$

Q.II Evaluate the following limits :

1. $\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

2. $\lim_{x\to 2}\left[\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

3. $\lim_{x\to 2}\left[\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}\right]$

4. $\lim_{y\to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

5. $= \lim_{x\to 0}\left(\frac{\sqrt{x^2+9}-\sqrt{2x^2+9}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\right)$

Q.III Evaluate the Following limits :

1. $\lim_{x\to 1}\left[\frac{x^2+x\sqrt{x}-2}{x-1}\right]$

2. $\lim_{x\to 0}\left[\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\right]$

3. $\lim_{x\to 4}\left[\frac{x^2+x-20}{\sqrt{3x+4}-4}\right]$

4. $\lim_{z\to 4}\left[\frac{3-\sqrt{5+z}}{1-\sqrt{5-z}}\right]$

5. $\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right)$

7.4 LIMIT OF A TRIGONOMETRIC FUNCTION :

Let's Learn :

To evaluate the limits involving trigonometric functions, we state -

1) $\lim_{x\to a} \sin x = \sin a$

2) $\lim_{x\to a} \cos x = \cos a$

Using these results and trigonometric identities, we solve some examples.

Evaluation of limits can be done by the method of Factorization, Rationalization or Simplification as the case may be. While solving examples based on trigonometric functions we can use trigonometric identities.

Squeeze theorem (Also known as **Sandwich theorem)**

Suppose $f(x)$, $g(x)$ and $h(x)$ are given functions such that $f(x) \le g(x) \le h(x)$ for all x in an open interval about a.

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$

So, $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)$

$\Rightarrow L \le \lim_{x\to a} g(x) \le L \quad \therefore \lim_{x\to a} g(x) = L$

7.4.2 Theorem : $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$; where θ is measured in radian.

Proof : First consider the case when θ is tending to zero through positive values.

We may take $0 < \theta < \frac{\pi}{2}$.

Draw a standard circle with radius r i.e. circles with centre at origin O and radius r.

Let A be the point of intersection of the circle and the X-axis. Take point P on the circle such that m∠AOP = θ

Draw PM ⊥ OX. Draw a line through A parallel to Y-axis to meet OP extended at B (fig. 7.2)

Area of ΔOAP < Area of sector OAP < Area of ΔOAB

$\therefore \frac{1}{2}\text{OA.PM} < \frac{1}{2}r^2\theta < \frac{1}{2}\text{OA.OB} \quad \text{..... (I)}$

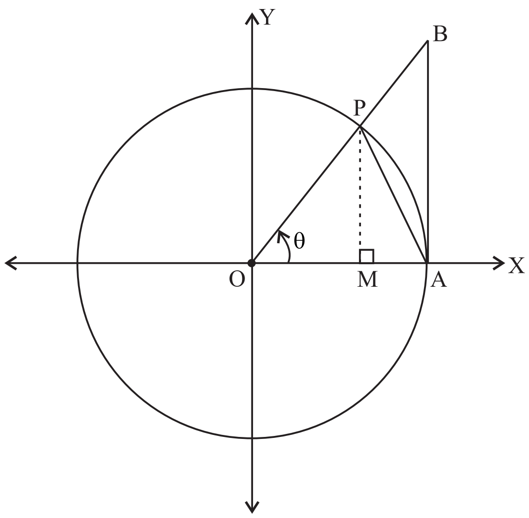

Fig. 7.3

In ΔOAP,

$$\sin\theta = \frac{PM}{OP}$$

$\therefore$ PM = OP sin θ

= r sin θ

Also, in ΔOAB

$$\tan\theta = \frac{AB}{OA}$$

$\therefore$ AB = OA tan θ

= r tan θ

using these in (I), we get

$$\frac{1}{2}r.r.\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r.r\tan\theta$$

i.e. $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ ... [Divide by $\frac{1}{2}r^2\sin\theta$]

$\therefore 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

i.e. $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

Taking limit as $\theta \to 0^+$

$\therefore \lim_{\theta\to 0^+} \cos\theta \le \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} \le \lim_{\theta\to 0^+} 1$

$\therefore 1 \le \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} \le 1$

By using squeeze theorem

$\therefore \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1 \quad \text{.... (II)}$

Now suppose θ→0 through negative values

Let θ = – ϕ where ϕ > 0. Also as θ → 0, ϕ→0

$\therefore \lim_{\theta\to 0^-} \dfrac{\sin\theta}{\theta} = \lim_{\phi\to 0^+} \dfrac{\sin(-\phi)}{-\phi}$

$= \lim_{\phi\to 0^+} \dfrac{-\sin\phi}{-\phi} = \lim_{\phi\to 0^+} \dfrac{\sin\phi}{\phi} = 1$

$\therefore \lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$ .... (III)

$\therefore$ from (II) and (III), $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Note:

Corollary 1 : $\lim_{\theta \to 0}\left(\frac{\theta}{\sin\theta}\right) = 1$

Corollary 2 : $\lim_{\theta \to 0}\left(\frac{\tan\theta}{\theta}\right) = 1$

Corollary 3 : $\lim_{\theta \to 0}\left(\frac{\theta}{\tan\theta}\right) = 1$

Corollary 4 : $\lim_{\theta \to 0}\left(\frac{\sin p\theta}{p\theta}\right) = 1$, ($p$ constant.)

Corollary 5 : $\lim_{\theta \to 0}\left(\frac{\tan p\theta}{p\theta}\right) = 1$, ($p$ constant.)

Corollary 6 : $\lim_{\theta \to 0}\left(\frac{p\theta}{\sin p\theta}\right) = 1$, ($p$ constant.)

Corollary 7 : $\lim_{\theta \to 0}\left(\frac{p\theta}{\tan p\theta}\right) = 1$, ($p$ constant.)

SOLVED EXAMPLES

Ex. 1) If $3x^2 + 2 \le f(x) \le 5x^2 - 6$ for all $x \in R$, then find $\lim_{x \to -2} f(x)$.

Solution : Let $g(x) = 3x^2 + 2$ and $h(x) = 5x^2 - 6$

So, we have $g(x) \le f(x) \le h(x)$

Taking as limit $x \to -2$ throughout we get

$$\lim_{x \to -2} g(x) \le \lim_{x \to -2} f(x) \le \lim_{x \to -2} h(x)$$

$$\lim_{x \to -2} (3x^2+2) \le \lim_{x \to -2} f(x) \le \lim_{x \to -2} (5x^2-6)$$

$$3(-2)^2 + 2 \le \lim_{x \to -2} f(x) \le 5(-2)^2 - 6$$

$$14 \le \lim_{x \to -2} f(x) \le 14$$

$\therefore \lim_{x \to -2} f(x) = 14$ .... [By squeeze theorem]

Ex. 2 : Evaluate : $\lim_{x \to \frac{\pi}{4}}\left(\frac{\cot x - 1}{\text{cosec}^2 x - 2}\right)$

Solution : $\lim_{x \to \frac{\pi}{4}}\left(\frac{\cot x - 1}{\text{cosec}^2 x - 2}\right)$

$$= \lim_{x \to \frac{\pi}{4}}\left(\frac{\cot x - 1}{\cot^2 x + 1 - 2}\right)$$

$$= \lim_{x \to \frac{\pi}{4}}\left(\frac{\cot x - 1}{\cot^2 x - 1}\right)$$

$$= \lim_{x \to \frac{\pi}{4}}\left(\frac{\cot x - 1}{(\cot x + 1)(\cot x - 1)}\right)$$

$$= \lim_{x \to \frac{\pi}{4}}\left(\frac{1}{(\cot x + 1)}\right)$$

.....[As x $\frac{\pi}{4}$ $\cot x - 1 \neq 0$]

$$= \frac{1}{\cot\left(\frac{\pi}{4}\right) + 1} = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Ex. 3) Evaluate : $\lim_{x \to \frac{3\pi}{2}}\left(\frac{\sqrt{3 - \sin x} - 2}{\cos^2 x}\right)$

Solution : $\lim_{x \to \frac{3\pi}{2}}\left(\frac{\sqrt{3 - \sin x} - 2}{\cos^2 x}\right)$

$$= \lim_{x \to \frac{3\pi}{2}}\left(\frac{\sqrt{3 - \sin x} - 2}{\cos^2 x} \times \frac{\sqrt{3 - \sin x} + 2}{\sqrt{3 - \sin x} + 2}\right)$$

$$= \lim_{x \to \frac{3\pi}{2}}\left(\frac{3 - \sin x - 4}{\cos^2 x\left(\sqrt{3 - \sin x} + 2\right)}\right)$$

$$= \lim_{x \to \frac{3\pi}{2}}\left(\frac{-1 - \sin x}{1 - \sin^2 x\left(\sqrt{3 - \sin x} + 2\right)}\right)$$

$$= \lim_{x \to \frac{3\pi}{2}}\left(\frac{-(1 + \sin x)}{(1 - \sin x)(1 + \sin x)\left(\sqrt{3 - \sin x} + 2\right)}\right)$$

.....As $x \to \frac{3\pi}{2}$, $\sin x \to -1$ and $1+\sin x \neq 0$

$$= \left(\frac{-1}{\left(1-\sin\left(\frac{3\pi}{2}\right)\right)\left(\sqrt{3-\sin\left(\frac{3\pi}{2}\right)}+2\right)} \right)$$

$$= \frac{-1}{(1+1)\left(\sqrt{3+1}+2\right)} = -\frac{1}{8}$$

Ex. 4. Evaluate : $\lim_{\theta\to 0}\left[\frac{\sin 7\theta}{\theta}\right]$

Solution : $\lim_{\theta\to 0}\left[\frac{\sin 7\theta}{\theta}\right]$

$$= \lim_{\theta\to 0}\left[\frac{\sin 7\theta}{7\theta}\right] \times 7$$

... as $\theta \to 0,\ 7\theta \to 0$

$$= 1 \times 7 \quad \left[\because \lim_{\theta\to 0}\frac{\sin p\theta}{p\theta} = 1\right]$$

$= 7$

Ex. 5. Evaluate : $\lim_{x\to 0}\left[\frac{\sin 8x}{\tan 4x}\right]$

Solution : $\lim_{x\to 0}\left[\frac{\sin 8x}{\tan 4x}\right]$

Divide Numerator and Denominator by x

$$= \lim_{x\to 0}\left[\frac{\frac{\sin 8x}{x}}{\frac{\tan 4x}{x}}\right]$$

$$= \frac{\lim_{x\to 0}\left[\frac{\sin 8x}{x}\right]}{\lim_{x\to 0}\left[\frac{\tan 4x}{x}\right]}$$

$$= \frac{\lim_{x\to 0}\left[\frac{\sin 8x}{8x}\times 8\right]}{\lim_{x\to 0}\left[\frac{\tan 4x}{4x}\times 4\right]}$$

$$= \frac{1\times 8}{1\times 4} = 2 \quad (\text{... as } x \to 0,\ 8x \to 0,\ 4x \to 0)$$

$$\text{....} \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right) = 1 \quad \lim_{\theta\to 0}\left(\frac{\tan p\theta}{p\theta}\right) = 1$$

Ex. 6. Evaluate : $\lim_{x\to 0}\left[\frac{2\sin x - \sin 2x}{x^3}\right]$

Solution : $\lim_{x\to 0}\left[\frac{2\sin x - \sin 2x}{x^3}\right]$

$$= \lim_{x\to 0}\left[\frac{2\sin x - 2\sin x.\cos x}{x^3}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sin x(1-\cos x)}{x.x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sin x}{x}\right]\times\lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{\sin x}{x}\right]\times\lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\times\frac{(1+\cos x)}{(1+\cos x)}\right]$$

$$= 2\times 1\times\lim_{x\to 0}\left[\frac{(1-\cos^2 x)}{x^2}\times\frac{1}{(1+\cos x)}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{(\sin^2 x)}{x^2}\right]\times\lim_{x\to 0}\left[\frac{1}{(1+\cos x)}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{\sin x}{x}\right]^2\times\lim_{x\to 0}\left[\frac{1}{(1+\cos 0)}\right]$$

$$= 2(1)^2\times\left[\frac{1}{1+1}\right] \quad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 2\times 1\times\frac{1}{2} = 1$$

Ex. 7. Evaluate : $\lim_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\right]$

Solution : $\lim_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\right]$

$$= \lim_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\times\frac{1+\cos x^2}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x^2(1-\cos^2 x^2)}{x^6}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x^2.\sin^2 x^2}{x^6}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin^3 x^2}{\left(x^2\right)^3}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x^2\to 0}\left[\frac{\sin x^2}{x^2}\right]^3\times\lim_{x\to 0}\left(\frac{1}{1+\cos x^2}\right)$$

.....[As $x\to 0$, $x^2\to 0$]

$$= (1)^3\frac{1}{1+\cos 0}=\frac{1}{1+1} = \frac{1}{2}$$

.....$\left[\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1, here\ \theta = x^2\right]$

Ex. 8. Evaluate : $\lim_{x\to 0}\left(\frac{\cos 5x^o - \cos 3x^o}{x^2}\right)$

Solution : $\lim_{x\to 0}\left(\frac{\cos 5x^o - \cos 3x^o}{x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{-2\sin 4x^o \sin x^o}{x^2}\right)$$

$$= -\ 2\lim_{x\to 0}\left(\frac{\sin 4x^o}{x}\times\frac{\sin x^o}{x}\right)$$

$$= -\ 2\lim_{x\to 0}\left(\frac{\sin 4\left(\frac{\pi x}{180}\right)}{x}\times\frac{\sin\left(\frac{\pi x}{180}\right)}{x}\right)$$

$$= -2\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{45}\right)}{x}\times\frac{\sin\left(\frac{\pi x}{180}\right)}{x}\right)$$

$$=-\ 2\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{45}\right)}{\frac{\pi x}{45}}\right)\times\frac{\pi}{45}\times\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}}\right)\times\frac{\pi}{180}$$

$$= -2\times(1)\times\frac{\pi}{45}\times(1)\times\frac{\pi}{180}......\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$$= -2\times 4\times\frac{\pi}{180}\times\frac{\pi}{180} = -8\left(\frac{\pi}{180}\right)^2$$

Activity-1 $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

Solution : $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

$$= \lim_{x\to 0}\left[\frac{\sin x\times\frac{1}{\square}-\sin x}{\sin^3 x}\right]$$

$$= \lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-\square\right)}{\sin^3 x}$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{\sin^2 x}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{1-\square}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{(\square)(1+\cos x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{\cos x}\times\frac{1}{(1+\cos x)}\right]$$

$$= \left[\frac{1}{\cos\square}\times\frac{1}{(1+\cos\square)}\right]$$

$$= \frac{1}{\square}\times\frac{1}{(1+\square)}$$

$$= \frac{1}{1}\times\frac{1}{\square}$$

$$= \frac{1}{\square}$$

EXERCISE 7.4

Q.I Evaluate the following limits :

1. $\lim_{\theta \to 0}\left[\frac{\sin(m\theta)}{\tan(n\theta)}\right]$

2. $\lim_{\theta \to 0}\left[\frac{1-\cos 2\theta}{\theta^2}\right]$

3. $\lim_{x \to 0}\left[\frac{x.\tan x}{1-\cos x}\right]$

4. $\lim_{x \to 0}\left(\frac{\sec x-1}{x^2}\right)$

Q.II Evaluate the Following limits :

1. $\lim_{x \to 0}\left[\frac{1-\cos(nx)}{1-\cos(mx)}\right]$

2. $\lim_{x \to \frac{\pi}{6}}\left[\frac{2-\operatorname{cosec} x}{\cot^2 x-3}\right]$

3. $\lim_{x \to \frac{\pi}{4}}\left[\frac{\cos x-\sin x}{\cos 2x}\right]$

Q.III Evaluate the following limits :

1. $\lim_{x \to 0}\left[\frac{\cos(ax)-\cos(bx)}{\cos(cx)-1}\right]$

2. $\lim_{x \to \pi}\left[\frac{\sqrt{1-\cos x}-\sqrt{2}}{\sin^2 x}\right]$

3. $\lim_{x \to \frac{\pi}{4}}\left[\frac{\tan^2 x-\cot^2 x}{\sec x-\operatorname{cosec} x}\right]$

4. $\lim_{x \to \frac{\pi}{6}}\left[\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}\right]$

Activity-1:

$$\lim_{x \to 0}\frac{1-\cos px}{x^2} = \frac{1}{2}p^2$$

Consider,

$$\lim_{x \to 0}\frac{1-\cos px}{x^2}$$

$$= \lim_{x \to 0}\frac{2.\square}{x^2}$$

Use $1-\cos A = 2\sin^2\frac{A}{2}$

$$= \lim_{x \to 0} 2.\left(\frac{\square}{x}\right)^2$$

$$= \lim_{x \to 0} 2\left(\frac{\sin\frac{px}{2}}{\frac{px}{2}}\times\square\right)^2$$

$$= \lim_{x \to 0} 2\left(\frac{\sin\frac{px}{2}}{\frac{px}{2}}\right)^2\left(\frac{p^2}{4}\right)$$

$$= 2\,(1)^2\left(\frac{p^2}{4}\right) \quad \because \lim_{x \to 0}\frac{\sin\theta}{\theta} = 1$$

$$= \square$$

$$\lim_{x \to 0}\frac{1-\cos px}{x^2} = \frac{1}{2}p^2$$

7.5 Substitution Method :

We will consider examples of trigonometric functions in which $x \to a$ where generally a takes the values such as $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ etc. In such a case we put $x - a = t$ so that as $x \to a$, $t \to o$.

SOLVED EXAMPLE

Ex. 1. Evaluate $\lim_{x\to\frac{\pi}{2}}\left[\frac{\cos x}{x-\frac{\pi}{2}}\right]$

Solution : Put $x-\frac{\pi}{2}=t \quad \therefore x=\frac{\pi}{2}+t$

As $x\to\frac{\pi}{2}$; $t\to 0$.

$$\lim_{x\to\frac{\pi}{2}}\left[\frac{\cos x}{x-\frac{\pi}{2}}\right]=\lim_{t\to 0}\left[\frac{\cos\left(\frac{\pi}{2}+t\right)}{t}\right]$$

$$=\lim_{t\to 0}\left[\frac{-\sin t}{t}\right]$$

$$=-\lim_{t\to 0}\left[\frac{\sin t}{t}\right]$$

$$=-1 \qquad \left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

Ex. 2. Evaluate $\lim_{x\to a}\left[\frac{\cos x-\cos a}{x-a}\right]$

Solution : Put $x-a=t \quad \therefore x=a+t$;

As $x\to a$, $t\to 0$.

$$\lim_{x\to a}\left[\frac{\cos x-\cos a}{x-a}\right]=\lim_{t\to 0}\left[\frac{\cos(a+t)-\cos a}{t}\right]$$

$$=\lim_{t\to 0}\left[\frac{-2\sin\left(\frac{2a+t}{2}\right)\cdot\sin\frac{t}{2}}{t}\right]$$

$$=2\lim_{t\to 0}\left[-\sin\left(a+\frac{t}{2}\right).\frac{\sin(t/2)}{t}\right]$$

$$=-2\lim_{t\to 0}\left[\sin\left(a+\frac{t}{2}\right)\right]\lim_{t\to 0}\left[\frac{\sin(t/2)}{t/2}\right]\left(\frac{1}{2}\right)$$

$$=-2\sin(a+0).(1)\frac{1}{2} \;\dots\; \left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$$=-\sin a$$

Ex. 3. Evaluate $\lim_{x\to 1}\left[\frac{1+\cos\pi x}{(1-x)^2}\right]$

Solution : Put $1-x=t \quad \therefore x=1-t$;

As $x\to 1$, $t\to 0$.

$$\lim_{x\to 1}\left[\frac{1+\cos\pi x}{(1-x)^2}\right]=\lim_{t\to 0}\left[\frac{1+\cos[\pi(1-t)]}{t^2}\right]$$

$$=\lim_{t\to 0}\left[\frac{1+\cos(\pi-\pi t)}{(1-x)^2}\right]$$

$$=\lim_{t\to 0}\left[\frac{1-\cos\pi t}{t^2}\right]$$

$$=\lim_{t\to 0}\left[\frac{2\sin^2\left(\frac{\pi t}{2}\right)}{t^2}\right]$$

$$=2\lim_{t\to 0}\left[\frac{\sin\left(\frac{\pi t}{2}\right)}{t}\right]^2$$

$$=2\lim_{t\to 0}\left[\frac{\sin\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}\right]^2\left(\frac{\pi}{2}\right)^2$$

$$=2(1)\left(\frac{\pi^2}{4}\right)\dots\left[\because \lim_{\theta\to 0}\left(\frac{\sin P\theta}{P\theta}\right)=1\right]$$

$$=\frac{\pi^2}{2}$$

Ex. 4. Evaluate $\lim_{x\to\frac{\pi}{3}}\left[\frac{\sqrt{3}-\tan x}{\pi-3x}\right]$

Solution : Put $\frac{\pi}{3}-x=t$, $\therefore x=\frac{\pi}{3}-t$,

$$\text{As } x\to\frac{\pi}{3},\ t\to 0$$

$$\lim_{x\to\frac{\pi}{3}}\left(\frac{\sqrt{3}-\tan x}{\pi-3x}\right)=\lim_{x\to\frac{\pi}{3}}\left(\frac{\sqrt{3}-\tan}{3\left(\frac{\pi}{3}-x\right)}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\tan\left(\frac{\pi}{3}-t\right)}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\frac{\tan(\pi/3)-\tan t}{1+\tan(\pi/3)\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\frac{\sqrt{3}-\tan t}{1+\sqrt{3}\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\frac{\sqrt{3}+3\tan t-\sqrt{3}+\tan t}{1+\sqrt{3}\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{4\tan t}{t(1+\sqrt{3}\tan t)}\right)$$

$$=\frac{4}{3}\lim_{t\to 0}\left(\frac{\tan t}{t}\right)\lim_{t\to 0}\left(\frac{1}{(1+\sqrt{3}\tan t)}\right)$$

$$=\frac{4}{3}(1)\times\frac{1}{\left(1+\sqrt{3}\tan 0\right)}\ \ldots\ldots\left[\because\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

$$=\frac{4}{3}$$

EXERCISE 7.5

I) Evaluate the following

1) $\lim_{x\to\frac{\pi}{2}}\left[\frac{\operatorname{cosec} x-1}{\left(\frac{\pi}{2}-x\right)^2}\right]$

2) $\lim_{x\to a}\frac{\sin x-\sin a}{\sqrt[5]{x}-\sqrt[5]{a}}$

3) $\lim_{x\to\pi}\left[\frac{\sqrt{5+\cos x}-2}{(\pi-x)^2}\right]$

4) $\lim_{x\to\frac{\pi}{6}}\left[\frac{\cos x-\sqrt{3}\sin x}{\pi-6x}\right]$

5) $\lim_{x\to 1}\left[\frac{1-x^2}{\sin\pi x}\right]$

II) Evaluate the following

1) $\lim_{x\to\frac{\pi}{6}}\left[\frac{2\sin x-1}{\pi-6x}\right]$

2) $\lim_{x\to\frac{\pi}{4}}\left[\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}\right]$

3) $\lim_{x\to\frac{\pi}{6}}\left[\frac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}\right]$

4) $\lim_{x \to a}\left[\frac{\sin\left(\sqrt{x}\right)-\sin\left(\sqrt{a}\right)}{x-a}\right]$

5) $\lim_{x \to \frac{\pi}{2}}\left[\frac{\cos 3x+3\cos x}{(2x-\pi)^3}\right]$

7.6 LIMITS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS :

Let's :Learn

We use the following results without proof.

1) $\lim_{x \to 0}\left(\frac{e^x-1}{x}\right) = \log e = 1$

2) $\lim_{x \to 0}\left(\frac{a^x-1}{x}\right) = \log a \;\; (a > 0,\ a \neq 0)$

3) $\lim_{x \to 0}[1+x]^{\frac{1}{x}} = e$

4) $\lim_{x \to 0}\left(\frac{\log(1+x)}{x}\right) = 1$

5) $\lim_{x \to 0}\left(\frac{e^{px}-1}{px}\right) = 1$, ($p$ constant)

6) $\lim_{x \to 0}\left(\frac{a^{px}-1}{px}\right) = \log a$, ($p$ constant)

7) $\lim_{x \to 0}\left(\frac{\log(1+px)}{px}\right) = 1$, ($p$ constant)

8) $\lim_{x \to 0}(1+px)^{\frac{1}{px}} = e$, ($p$ constant)

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x \to 0}\left[\frac{5^x-1}{\sin x}\right]$

Solution : $\lim_{x \to 0}\left[\frac{5^x-1}{\sin x}\right]$

Divide Numerator and Denominator by x

$$= \lim_{x \to 0}\left[\frac{\frac{5^x-1}{x}}{\frac{\sin x}{x}}\right]$$

$$= \frac{\lim_{x \to 0}\left[\frac{5^x-1}{x}\right]}{\lim_{x \to 0}\left[\frac{\sin x}{x}\right]}$$

$$= \frac{\log 5}{1}$$

$$\ldots\left[\because \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)=1, \lim_{x \to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

$$= (\log 5)$$

Ex. 2. Evaluate : $\lim_{x \to 0}\left[\frac{5^x - 3^x}{x}\right]$

Solution : Given $\lim_{x \to 0}\left[\frac{5^x-3^x}{x}\right]$

$$= \lim_{x \to 0}\left[\frac{5^x-1-3^x+1}{x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\left(5^x-1\right)-\left(3^x-1\right)}{x}\right]$$

$$= \lim_{x \to 0}\frac{\left(5^x-1\right)}{x} - \lim_{x \to 0}\frac{\left(3^x-1\right)}{x}$$

$= \log 5 - \log 3 \quad ..... \lim_{x\to a}\left(\frac{a^x - 1}{x}\right) = \log a$

$= log\left(\frac{5}{3}\right)$

Ex. 3. Evaluate : $\lim_{x\to 0}\left[1+\frac{5x}{6}\right]^{\frac{1}{x}}$

Solution : $\lim_{x\to 0}\left[1+\frac{5x}{6}\right]^{\frac{1}{x}}$

$= \lim_{x\to 0}\left[\left(1+\frac{5x}{6}\right)^{\frac{1}{\frac{6}{5x}}}\right]^{\frac{5}{6}}$

$= (e)^{\frac{5}{6}}$

Ex. 4. Evaluate : $\lim_{x\to 0}\left[\frac{3x+2}{2-5x}\right]^{\frac{1}{3x}}$

Solution: $\lim_{x\to 0}\left[\frac{3x+2}{2-5x}\right]^{\frac{1}{3x}} = \lim_{x\to 0}\left[\frac{2+3x}{2-5x}\right]^{\frac{1}{3x}}$

$= \lim_{x\to 0}\left[\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)}\right]^{\frac{1}{3x}}$

$= \lim_{x\to 0}\left[\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}}\right]^{\frac{1}{3}}$

$= \frac{\left[\lim_{x\to 0}\left(1+\frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}\times\frac{1}{3}}}{\left[\lim_{x\to 0}\left(1+\frac{-5x}{2}\right)^{\frac{2}{-5x}}\right]^{\frac{-5}{2}\times\frac{1}{3}}}$

$= \frac{e^{\frac{3}{6}}}{e^{\frac{-5}{6}}} \; ..... \left[\because \lim_{x\to 0}(1+kx)^{\frac{1}{kx}} = e\right]$

$= e^{\frac{8}{6}} = e^{\frac{4}{3}}$

Ex. 5. Evaluate : $\lim_{x\to 0}\left[\frac{\log 4 + \log(0.25+x)}{x}\right]$

Solution : $\lim_{x\to 0}\left[\frac{\log 4 + \log(0.25+x)}{x}\right]$

$= \lim_{x\to 0}\left[\frac{\log[4(0.25+x)]}{x}\right]$

$= \lim_{x\to 0}\left[\frac{\log(1+4x)}{x}\right]$

$= 4\times\lim_{x\to 0}\left[\frac{\log(1+4x)}{4x}\right]$

$= 4(1) \; ..... \left[\because \lim_{x\to 0}\left[\frac{\log(1+px)}{px}\right] = 1\right]$

$= 4$

Ex. 6. Evaluate : $\lim_{x\to 0}\left(\frac{e^{2x}+e^{-2x}-2}{x\sin x}\right)$

Solution :

$\lim_{x\to 0}\left(\frac{e^{2x}+e^{-2x}-2}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{e^{2x}\left(e^{2x}+e^{-2x}-2\right)}{e^{2x}x\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{4x}+1-2e^{2x}}{e^{2x}x\sin x}\right)$

$$= \lim_{x\to 0}\left(\frac{(e^{2x})^2 - 2e^{2x} + 1}{e^{2x} x \sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(e^{2x}-1)^2}{x\sin x} \times \frac{1}{e^{2x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{(e^{2x}-1)^2}{x^2}}{\frac{x\sin x}{x^2}} \times \frac{1}{e^{2x}}\right)$$

....[As $x\to 0$, $x \neq 0$, $x^2 \neq 0$]

$$= \frac{\left[\lim_{x\to 0}\left(\frac{e^{2x}-1}{2x}\right)\right]^2 \times 4}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} \times \lim_{x\to 0}\left(\frac{1}{e^{2x}}\right)$$

$$= \frac{(1)^2 \times 4}{1} \times \frac{1}{e^o} \ldots\ldots \left[\because \lim_{x\to 0}\frac{e^{kx}-1}{kx} = 1, \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

= 4

Ex. 7. Evaluate : $\lim_{x\to 0}\left(\frac{21^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$

Solution : $\lim_{x\to 0}\left(\frac{21^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$

$$= \lim_{x\to 0}\left(\frac{7^x.3^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{7^x\left(3^x-1\right) - \left(3^x-1\right)}{x\log(1+x)}\right)$$

$$= \lim_{x\to 0}\left[\frac{\left(3^x-1\right)\left(7^x-1\right)}{x\log(1+x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{\left(3^x-1\right)\left(7^x-1\right)}{x^2}}{\frac{x\log(1+x)}{x^2}}\right] \ldots[As\ x\to 0, x^2 \neq 0]$$

$$= \lim_{x\to 0}\left[\frac{\frac{3^x-1}{x}\times\frac{7^x-1}{x}}{\frac{\log(1+x)}{x}}\right]$$

$$= \frac{\log 3.\log 7}{1}$$

$$\ldots\left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log a, \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$$= \log 3.\log 7$$

Activity-3

Evaluate : $\lim_{x\to 0}\left[\frac{8^x - 4^x - 2^x + 1}{x^2}\right]$

$$= \lim_{x\to 0}\left[\frac{(4\times\square)^x - 4^x - 2^x + 1}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(4^x\times\square)^x - 4^x - 2^x + 1}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{4^x(\square - 1) - (2^x - 1)}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2^x-1).\ \ (4^x-1)}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2^x-1)}{x}\right] \times \lim_{x\to 0}\left[\frac{(4^x-1)}{x}\right]$$

$$= \square\ \square$$

Activity-4:

Evaluate : $\lim_{x\to 0}\left[\frac{e^x - \sin x - 1}{x}\right]$

$= \lim_{x\to 0}\left[\frac{(e^x - 1) - \square}{x}\right]$

$= \lim_{x\to 0}\left[\frac{\square}{x} - \frac{\sin x}{\square}\right]$

$= \lim_{x\to 0}\left[\frac{\square}{x}\right] - \lim_{x\to 0}\left[\frac{\sin x}{x}\right]$

$= \square - 1$

$= 1 - 1$

$= \square$

EXERCISE 7.6

Q.I Evaluate the following limits :

1) $\lim_{x\to 0}\left[\frac{9^x - 5^x}{4^x - 1}\right]$

2) $\lim_{x\to 0}\left[\frac{5^x + 3^x - 2^x - 1}{x}\right]$

3) $\lim_{x\to 0}\left(\frac{a^x + b^x + c^x - 3}{\sin x}\right)$

4) $\lim_{x\to 0}\left(\frac{6^x + 5^x + 4^x - 3^{x+1}}{\sin x}\right)$

5) $\lim_{x\to 0}\left(\frac{8^{\sin x} - 2^{\tan x}}{e^{2x} - 1}\right)$

Q.II Evaluate the following limits :

1) $\lim_{x\to 0}\left[\frac{3^x + 3^{-x} - 2}{x.\tan x}\right]$

2) $\lim_{x\to 0}\left[\frac{3+x}{3-x}\right]^{\frac{1}{x}}$

3) $\lim_{x\to 0}\left[\frac{5x+3}{3-2x}\right]^{\frac{2}{x}}$

4) $\lim_{x\to 0}\left[\frac{\log(3-x) - \log(3+x)}{x}\right]$

5) $\lim_{x\to 0}\left[\frac{4x+1}{1-4x}\right]^{\frac{1}{x}}$

6) $\lim_{x\to 0}\left[\frac{5+7x}{5-3x}\right]^{\frac{1}{3x}}$

Q.III Evaluate the following limits :

1) $\lim_{x\to 0}\left[\frac{a^x - b^x}{\sin(4x) - \sin(2x)}\right]$

2) $\lim_{x\to 0}\left[\frac{(2^x - 1)^3}{(3^x - 1).\sin x.\log(1+x)}\right]$

3) $\lim_{x\to 0}\left[\frac{15^x - 5^x - 3^x + 1}{x\ .\ \sin x}\right]$

4) $\lim_{x\to 0}\left[\frac{(25)^x - 2(5)^x + 1}{x\ .\ \sin x}\right]$

5) $\lim_{x\to 0}\left[\frac{(49)^x - 2(35)^x + (25)^x}{\sin x.\log(1+2x)}\right]$

7.7 LIMIT AT INFINITY : (FUNCTION TENDING TO INFINITY)

Let's :Learn

7.7.1 Limit at infinity :

Let us consider the function $f(x) = \frac{1}{x}$

Observe that as x approaches to ∞ or $-\infty$ the value of $f(x)$ is shown below,

i) Observe the following table for $f(x) = \frac{1}{x}$

x	1	10	100	1000	10000	100000	...
$f(x)$	1	0.1	0.01	0.001	0.0001	0.00001	...

We see that as x assumes larger and larger values, $\frac{1}{x}$ assumes the value nearer and nearer to zero.

$$\therefore \lim_{x \to \infty} \frac{1}{x} = 0$$

Definition : A function f is said to tend to limit 'l' as x tends to ∞ if for given $\in > 0$, there exists a positive number M such that $|f(x) - l| < \in$, $\forall$ x in the domain of f for which $x > M$

$$\therefore \lim_{x \to \infty} f(x) = l$$

ii) Observe the following table for $f(x) = \frac{1}{x}$

x	-1	-10	-100	-1000	-10000	-100000	...
$f(x)$	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001	...

We see that as x assumes values which tend to $-\infty$, $\frac{1}{x}$ assumes the value nearer and nearer to zero.

$$\therefore \lim_{x \to -\infty} \frac{1}{x} = 0$$

Definition : A function f is said to tend to limit 'l' as x tends to $-\infty$ if for given $\in > 0$, there exists a positive number M such that $|f(x) - l| < \in$, for all $x > M$

$$\therefore \lim_{x \to -\infty} f(x) = l$$

Note : Whenever expression is of the form $\frac{\infty}{\infty}$, then divide, by suitable power of x to get finite limits of numerator as well as denominator.

7.7.2 Infinite Limits :

Let us consider the function $f(x) = \frac{1}{x}$.

Observe the behavior of $f(x)$ as x approaches zero from right and from left.

i) Observe the following table for $f(x) = \frac{1}{x}$

$x =$	1	0.1	0.01	0.001	0.0001	0.00001	...
$f(x)$	1	10	100	1000	10000	100000	...

We see that as x assumes values nearer 0, but greater than 0, $\frac{1}{x}$ assumes the values larger and larger.

$$\therefore \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} \to \infty$$

ii) Observe the following table for $f(x) = \frac{1}{x}$

$x =$	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001	...
$f(x)$	-1	-10	-100	-1000	-10000	-100000	...

We see that as x assumes values nearer to 0, but less than 0, $\frac{1}{x}$ assumes the values which tends to $-\infty$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} \to -\infty$$

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x\to\infty}\left[\frac{ax+b}{cx+d}\right]$

Solution : $\lim_{x\to\infty}\left[\frac{ax+b}{cx+d}\right]$

$$= \lim_{x\to\infty}\left[\frac{\frac{ax+b}{x}}{\frac{cx+d}{x}}\right]$$

$$= \frac{\lim_{x\to\infty}\left[a+\frac{b}{x}\right]}{\lim_{x\to\infty}\left[c+\frac{d}{x}\right]}$$

$$= \frac{a+0}{c+0} \quad \text{--- as } x\to\infty,\ \frac{1}{x}\to 0$$

$$= \frac{a}{c}$$

Ex. 2. Evaluate : $\lim_{x\to\infty}\left[\frac{10x^2+5x+3}{5x^2-3x+8}\right]$

Solution : $\lim_{x\to\infty}\left[\frac{10x^2+5x+3}{5x^2-3x+8}\right]$

Divide by x^2 to get finite limits of the numerator as well as of the denominator,

$$= \lim_{x\to\infty}\left[\frac{\frac{10x^2+5x+3}{x^2}}{\frac{5x^2-3x+8}{x^2}}\right]$$

$$= \frac{\lim_{x\to\infty}\left[10+\frac{5}{x}+\frac{3}{x^2}\right]}{\lim_{x\to\infty}\left[5-\frac{3}{x}+\frac{8}{x^2}\right]}$$

$$= \frac{10+0+0}{5-0+0} \qquad \left(\because \lim_{x\to\infty}\frac{1}{x}=0\right)$$

$$= \frac{10}{5}$$

$$= 2$$

Ex. 3. Evaluate : $\lim_{x\to\infty}\left[\sqrt{x^2+3x}-x\right]$

Solution : $\lim_{x\to\infty}\left[\sqrt{x^2+3x}-x\right]$

$$= \lim_{x\to\infty}\left[\frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{(\sqrt{x^2+3x}+x)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}\right]$$

$$= \lim_{x\to\infty}\left[\frac{3x}{\sqrt{x^2+3x}+x}\right]$$

$$= \lim_{x\to\infty}\left[\frac{3x}{x\sqrt{1+\frac{3}{x}}+x}\right]$$

$$= \lim_{x\to\infty}\left[\frac{3}{\sqrt{1+\frac{3}{x}}+1}\right]$$

$$= \frac{3}{\sqrt{1+0}+1} \qquad \left(\because \lim_{x\to\infty}\frac{1}{x}=0\right)$$

$$= \frac{3}{1+1}$$

$$= \frac{3}{2}$$

EXERCISE 7.7

I Evaluate the following :

1) $\lim_{x\to\infty}\left[\frac{ax^3+bx^2+cx+d}{ex^3+fx^2+gx+h}\right]$

2) $\lim_{x\to\infty}\left[\frac{x^3+3x+2}{(x+4)(x-6)(x-3)}\right]$

3) $\lim_{x\to\infty}\left[\frac{7x^2+5x-3}{8x^2-2x+7}\right]$

II Evaluate the following :

1) $\lim_{x\to\infty}\left[\frac{7x^2+2x-3}{\sqrt{x^4+x+2}}\right]$

2) $\lim_{x\to\infty}\left[\sqrt{x^2+4x+16}-\sqrt{x^2+16}\right]$

3) $\lim_{x\to\infty}\left[\sqrt{x^4+4x^2}-x^2\right]$

III Evaluate the following :

1) $\lim_{x\to\infty}\left[\frac{(3x^2+4)(4x^2-6)(5x^2+2)}{4x^6+2x^4-1}\right]$

2) $\lim_{x\to\infty}\left[\frac{(3x-4)^3(4x+3)^4}{(3x+2)^7}\right]$

3) $\lim_{x\to\infty}\left[\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)\right]$

4) $\lim_{x\to\infty}\left[\frac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}}\right]$

5) $\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

Let's Remember

Note : For limits of trigonometric functions, angle is supposed to be in radian.

1) $\lim_{x\to 0}\frac{\sin x}{x}=1$

2) $\lim_{x\to a}\sin x=\sin a$

3) $\lim_{x\to 0}\cos x=\cos a$

4) $\lim_{x\to 0}\frac{\tan x}{x}=1$

5) $\lim_{x\to 0}\frac{\sin kx}{x}=k$

6) $\lim_{x\to 0}\frac{\tan kx}{x}=k$

7) $\lim_{x\to 0}x.\sin\left(\frac{1}{x}\right)=0$

8) $\lim_{x\to 0}\frac{1-\cos p x}{x^2}=\frac{p^2}{2}$

9) $\lim_{x\to 0}\frac{\cos m x-\cos n x}{x^2}=\frac{n^2-m^2}{2}$

10) $\lim_{x\to 0}\frac{a^x-1}{x}=\log a,\ \forall\ a>0$

11) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

12) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

13) $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)=na^{n-1}$ for $a>0$

14) $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1$

15) $\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$

16) $\lim_{x\to\infty}\left(\frac{k}{x^p}\right) = 0$ for k, p $\in$ R and p > 0

17) As $x \to 0, \left(\frac{1}{x}\right) \to \infty$

18) $\lim_{x\to\infty}\left(\frac{a}{b}\right)^x = 0$, if $a < b$

MISCELLANEOUS EXERCISE - 7

I) Select the correct answer from the given alternatives.

1) $\lim_{x\to 2}\left(\frac{x^4-16}{x^2-5x+6}\right) =$

A) 23 B) 32 C) –32 D) –16

2) $\lim_{x\to -2}\left(\frac{x^7+128}{x^3+8}\right) =$

A) $\frac{56}{3}$ B) $\frac{112}{3}$ C) $\frac{121}{3}$ D) $\frac{28}{3}$

3) $\lim_{x\to 3}\left(\frac{1}{x^2-11x+24}+\frac{1}{x^2-x-6}\right) =$

A) $-\frac{2}{25}$ B) $\frac{2}{25}$ C) $\frac{7}{25}$ D) $-\frac{7}{25}$

4) $\lim_{x\to 5}\left(\frac{\sqrt{x+4}-3}{\sqrt{3x-11}-2}\right) =$

A) $\frac{-2}{9}$ B) $\frac{2}{7}$ C) $\frac{5}{9}$ D) $\frac{2}{9}$

5) $\lim_{x\to \frac{\pi}{3}}\left(\frac{\tan^2 x-3}{\sec^3 x-8}\right) =$

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{4}$

6) $\lim_{x\to 0}\left(\frac{5\sin x - x\cos x}{2\tan x - 3x^2}\right) =$

A) 0 B) 1 C) 2 D) 3

7) $\lim_{x\to \frac{\pi}{2}}\left[\frac{3\cos x+\cos 3x}{(2x-\pi)^3}\right] =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) $\frac{1}{4}$

8) $\lim_{x\to 0}\left(\frac{15^x-3^x-5^x+1}{\sin^2 x}\right) =$

A) log 15 B) log 3 + log 5

C) log 3.log 5 D) 3 log 5

9) $\lim_{x\to 0}\left(\frac{3+5x}{3-4x}\right)^{\frac{1}{x}} =$

A) e^3 B) e^6 C) e^9 D) e^{-3}

10) $\lim_{x\to 0}\left[\frac{\log(5+x)-\log(5-x)}{\sin x}\right] =$

A) $\frac{3}{2}$ B) $-\frac{5}{2}$ C) $-\frac{1}{2}$ D) $\frac{2}{5}$

11) $\lim_{x\to \frac{\pi}{2}}\left(\frac{3^{\cos x}-1}{\frac{\pi}{2}-x}\right) =$

A) 1 B) log 3 C) $3^{\frac{\pi}{2}}$ D) 3 log 3

12) $\lim_{x\to 0}\left[\frac{x.\log(1+3x)}{(e^{3x}-1)^2}\right] =$

A) $\frac{1}{e^9}$ B) $\frac{1}{e^3}$ C) $\frac{1}{9}$ D) $\frac{1}{3}$

13) $\lim_{x\to 0}\left[\frac{(3^{\sin x}-1)^3}{(3^x-1)\cdot\tan x\cdot\log(1+x)}\right] =$

A) 3 log 3 B) 2 log 3

C) $(\log 3)^2$ D) $(\log 3)^3$

14) $\lim_{x\to 3}\left[\frac{5^{x-3}-4^{x-3}}{\sin(x-3)}\right] =$

A) log 5 – 4 B) log $\frac{5}{4}$

C) $\frac{\log 5}{\log 4}$ D) $\frac{\log 5}{4}$

15) $\lim_{x\to\infty}\left[\frac{(2x+3)^7\,(x-5)^3}{(2x-5)^{10}}\right] =$

A) $\frac{3}{8}$ B) $\frac{1}{8}$ C) $\frac{1}{6}$ D) $\frac{1}{4}$

II) Evaluate the following.

1) $\lim_{x\to0}\left[\frac{(1-x)^5-1}{(1-x)^3-1}\right]$

2) $\lim_{x\to0}[x]$ ([*] is a greatest integer function.)

3) If $f(r)=\pi r^2$ then find $\lim_{h\to0}\left[\frac{f(r+h)-f(r)}{h}\right]$

4) $\lim_{x\to0}\left[\frac{x}{|x|+x^2}\right]$

5) Find the limit of the function, if it exists, at $x=1$

$f(x)=\begin{cases}7-4x & for \quad x<1\\ x^2+2 & for \quad x\ge1\end{cases}$

6) Given that $7x\le f(x)\le 3x^2-6$ for all x. Determine the value of $\lim_{x\to3}f(x)$

7) $\lim_{x\to0}\left[\frac{secx^2-1}{x^4}\right]$

8) $\lim_{x\to0}\left[\frac{e^x+e^{-x}-2}{x.tanx}\right]$

9) $\lim_{x\to0}\left[\frac{x(6^x-3^x)}{\cos(6x)-\cos(4x)}\right]$

10) $\lim_{x\to0}\left[\frac{a^{3x}-a^{2x}-a^x+1}{x.tanx}\right]$

11) $\lim_{x\to a}\left[\frac{sinx-sina}{x-a}\right]$

12) $\lim_{x\to2}\left[\frac{\log x-\log 2}{x-2}\right]$

13) $\lim_{x\to1}\left[\frac{ab^x-a^xb}{x^2-1}\right]$

14) $\lim_{x\to0}\left[\frac{(5^x-1)^2}{(2^x-1)\log(1+x)}\right]$

15) $\lim_{x\to\infty}\left[\frac{(2x+1)^2(7x-3)^3}{(5x+2)^5}\right]$

16) $\lim_{x\to a}\left[\frac{x\cos a-a\cos x}{x-a}\right]$

17) $\lim_{x\to\frac{\pi}{4}}\left[\frac{(\sin x-\cos x)^2}{\sqrt{2}-\sin x-\cos x}\right]$

18) $\lim_{x\to1}\left[\frac{2^{2x-2}-2^x+1}{\sin^2(x-1)}\right]$

19) $\lim_{x\to1}\left[\frac{4^{x-1}-2^x+1}{(x-1)^2}\right]$

20) $\lim_{x\to1}\left[\frac{\sqrt{x}-1}{\log x}\right]$

21) $\lim_{x\to0}\left(\frac{\sqrt{1-\cos x}}{x}\right)$

22) $\lim_{x\to1}\left(\frac{x+3x^2+5x^3+\cdots\cdots+(2n-1)x^n-n^2}{x-1}\right)$

23) $\lim_{x\to0}$

$$\left\{\frac{1}{x^{12}}\left[1-\cos\left(\frac{x^2}{2}\right)-\cos\left(\frac{x^4}{4}\right)+\cos\left(\frac{x^2}{2}\right)\cos\left(\frac{x^4}{4}\right)\right]\right\}$$

24) $\lim_{x\to\infty}\left(\frac{8x^2+5x+3}{2x^2-7x-5}\right)^{\frac{4x+3}{8x-1}}$